

Hausdorff dimension of wiggly metric spaces

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Abstract

We answer a question of Bishop and Tyson and generalize a result of Bishop and Jones by showing that a compact connected metric space that is far from any geodesic uniformly at all scales and locations has dimension strictly greater than one quantitatively.

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1 Introduction

1.1 Background and Main Results

The archetypal example of a connected fractal set is the von Koch snowflake lying in the plane. Its Hausdorff dimension can be easily computed, although the fact that it should have dimension larger than one can be gleaned from much simpler geometric information than its self-similar structure: all that is required for a connected subset of Euclidean space is that it is uniformly oscillatory or “wiggly” at every scale and location.

To make this more precise, we recall the Jones β -numbers: for a subset K of a Hilbert space \mathcal{H} , we define for $x \in K$ and $r > 0$

$$\beta(x, r) = \beta_K(x, r) = \frac{1}{r} \inf_L \sup \{\text{dist}(x, L) : x \in K \cap B(x, r)\} \quad (1.1)$$

where the infimum is taken over all lines $L \subseteq \mathcal{H}$ (we drop the K when there is no confusion about what set we’re talking about).

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Theorem 1. ([1, Theorem 1.1]) *There is a constant $c > 0$ such that the following holds. Let $\Gamma \subseteq \mathbb{R}^2$ and suppose that there is $r_0 > 0$ such that for all $r \in (0, r_0)$ and all $x \in \Gamma$, $\beta_\Gamma(x, r) > \beta_0$. Then the Hausdorff dimension¹ of Γ satisfies $\dim \Gamma \geq 1 + c\beta_0^2$.*

Theorem 1 roughly states that any connected set that deviates from lying near a line or is uniformly non-flat at each scale and location must necessarily have large dimension². Here, the β -number serves as an appropriate measure of the degree of oscillation or non-flatness in your set.

Our goal is to generalize this result to the metric space setting (that is, for a metric curve), but before stating the main results, we discuss the techniques and steps involved in proving Theorem 1 to elucidate why the techniques don't immediately carry over, and to discuss how to adapt them to our setting.

The main tool in proving Theorem 1 is the *Analyst's Traveling Salesman Theorem*, which we state below.

Theorem 2. ([13, Theorem 1.1]) *For a set K in a Hilbert space \mathcal{H} , let $A > 1$ and X_n be a nested sequence of maximal 2^{-n} -nets.*

$$\beta^K := \text{diam} K + \sum_{n \in \mathbb{Z}} \sum_{x \in X_n} \beta_K^2(x, A2^{-n})2^{-n}. \quad (1.2)$$

There is A_0 such that for $A > A_0$ and any set K , if β^K is finite, then K may be contained in a connected set Γ such that

$$\mathcal{H}^1(\Gamma) \leq C_A \beta^K$$

for some C_A depending only on A . Moreover, if Γ is any rectifiable set of finite length, then

$$\beta^\Gamma \leq C_A \mathcal{H}^1(\Gamma) \quad (1.3)$$

for any $A > 1$.

At the time of [1], this was only known for $\mathcal{H} = \mathbb{R}^2$ and was originally due to Jones [7]. This was subsequently generalized to \mathbb{R}^n by Okikiolu [10] and then to Hilbert space by Schul [13].

¹See Section 2 for the definition of Hausdorff dimension and other definitions and notation.

²We quickly remark that there are analogues of Theorem 1 for surfaces of higher topological dimension, see for example [5].

The proof of Theorem 1 goes roughly as follows: one constructs a *Frostmann measure* μ satisfying

$$\mu(B(x, r)) \leq Cr^s \quad (1.4)$$

for some $C > 0$, $s = 1 + c\beta_0^2$ and for all $x \in \Gamma$ and $r > 0$. This easily implies that the Hausdorff dimension of Γ is at least s (see [9, Theorem 8.8] and that section for a discussion on Frostmann measures). They build such a measure on Γ inductively by deciding the values $\frac{\mu(Q_n)}{\mu(Q)}$ for each dyadic cube Q intersecting Γ and for each n -th generation descendant Q_n intersecting Γ , where n is some large number that will depend on β_0 . If the number of such n -th generation descendants is large enough, we can choose the ratios and hence disseminate the mass $\mu(Q)$ amongs the descendants Q_n in such a way that the ratios will be very small and (1.4) will be satisfied. To show that there are enough descendants, one looks at the skeletons of the n -th generation descendants of Q and uses the second half of Theorem 2 coupled with the non-flatness condition in the statement of Theorem 1 to guarantee that the total length of this skeleton (and hence the number of cubes) will be large.

In the metric space setting, however, no such complete analogue of Theorem 2 exists, and it is not even clear what the appropriate analogue of a β -number should be. In [6], Hahlomaa gives a good candidate for a β -number in metric space using Menger curvature and uses it to show that if the sum in (1.2) is finite for a metric space K (using his definition of β_K), then it can be contained in the Lipschitz image of a subset of the real line (analogous to the first half of Theorem 2). An example of Schul [12], however, shows that the converse of Theorem 2 is false in general that (1.3) with Hahlomaa's β_K does not hold with the same constant for all curves in ℓ^1 . We refer to [12] for a good summary on the Analyst's Traveling Salesman Problem.

In generalizing Theorem 1, we still make use of β -type quantities and techniques, although the β -number that we use differs from both Jones and Hahlomaa's definitions, and instead is inspired by one defined by Bishop and Tyson. In [2], they suggest the following quantity for measuring not flatness, but the deviation from a geodesic in a metric space. If (X, d) is a metric space, $B_X(x, r) = \{y \in X : d(x, y) < r\}$, and $y_0, \dots, y_n \in B_X(x, r)$ an ordered sequence, define

$$\partial(y_0, \dots, y_n) = \sum_{i=0}^{n-1} |y_i - y_{i+1}| - |y_0 - y_n| + \sup_{z \in B_X(x, r)} \min_{i=1, \dots, n} |z - y_i| \quad (1.5)$$

and define

$$\hat{\beta}_X(x, r) = \inf_{\{y_i\} \subseteq B_X(x, r)} \frac{\partial(y_0, \dots, y_n)}{|y_0 - y_n|} \quad (1.6)$$

where the infimum is over all finite ordered sequences in $B_X(x, r)$ of any length n .

In [2], Bishop and Tyson ask whether, for a compact connected metric space Γ , (1.6) being uniformly larger than zero is enough to guarantee that $\dim X > 1$. We answer this in the affirmative.

Theorem 3. *There is $\kappa > 0$ such that the following holds. If Γ is a compact connected metric space and $\hat{\beta}_\Gamma(x, r) > \beta > 0$ for all $x \in \Gamma$ and $r \in (0, r_0)$ for some $r_0 > 0$, then $\dim \Gamma \geq 1 + \kappa\beta^4$.*

Instead of working with $\hat{\beta}$, we will work with a different quantity. First, by Kuratowski embedding theorem, we may assume X is a subset of ℓ^∞ , whose norm we denote by $|\cdot|$. Let $B(x, r) = B_{\ell^\infty}(x, r)$ and define

$$\beta'_X(x, r) = \inf_s \frac{\ell(s) - |s(0) - s(1)| + \sup_{z \in B(x, r)} d(z, s([0, 1]))}{|s(0) - s(1)|} \quad (1.7)$$

the infimum is over all curves $s : [0, 1] \rightarrow B(x, r) \subseteq \ell^\infty$ and

$$\ell(s) = \sup_{\{t_i\}_{i=0}^n} \sum_{i=0}^{n-1} |s(t_i) - s(t_{i+1})|$$

where the supremum is over all partitions $0 = t_0 < t_1 < \dots < t_n = 1$. In general, if s is a function defined on a union of disjoint open intervals $\{I_j\}_{j=1}^\infty$, we define

$$\ell(s|_{\cup I_j}) = \sum_j \ell(s|_{I_j}).$$

Observe that if s is just a straight line segment through the center of the ball with length $2r$, this gives $\beta'_X(x, r) \leq \frac{1}{2}$.

The quantity $\beta'(x, r)$ measures in a sense how well $X \cap B(x, r)$ may be approximated by a geodesic. To see this, note that if the $\frac{\beta'(x, r)}{2}|s(0) - s(1)|$ neighborhood of $s([0, 1])$ contains $X \cap (x, r)$, then its length must be at least $(1 + \frac{\beta'(x, r)}{2})|s(0) - s(1)|$, or $\frac{\beta'(x, r)}{2}|s(0) - s(1)|$ more than if it just took a geodesic between $s(0)$ and $s(1)$. The quantity $\hat{\beta}$ similarly measures how well the portion of X may be approximated by a geodesic polygonal path with vertices in X . In Figure 1, we compare the meanings of β , $\hat{\beta}$, and β' .

We will refer to the quantities $\ell(s)$ and $\partial(y_0, \dots, y_n)$ as the *geodesic deviation* of s and $\{y_0, \dots, y_n\}$ respectively. We will also say $\hat{\beta}_X(x, r)$ and $\beta'_X(x, r)$ measure the *geodesic deviation* of X inside the ball $B(x, r)$.

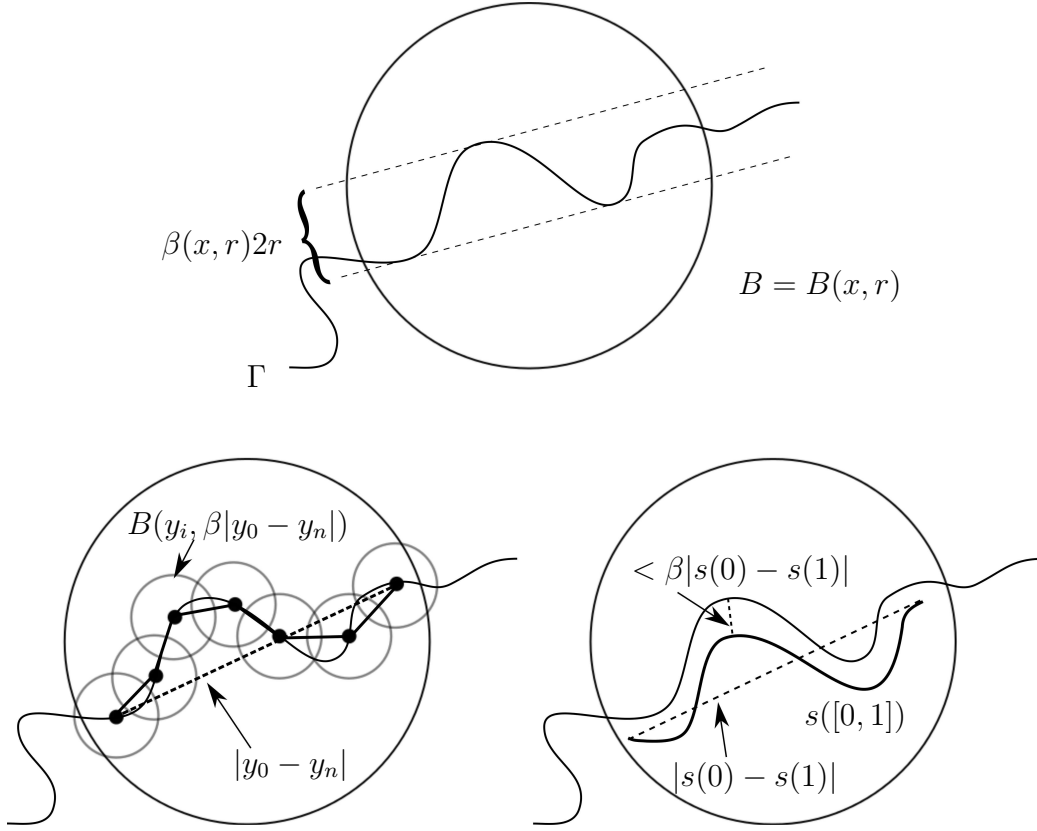


Figure 1: In each of the three figures above is a ball $B = B(x, r)$ containing a portion of a curve Γ . In the first picture, $\beta(x, r)2r$ is the width of the smallest tube containing $\Gamma \cap B(x, r)$. In the second, we see that $\hat{\beta}(x, r)$ is such that for $\beta > \hat{\beta}(x, r)$, there are $y_0, \dots, y_n \in \Gamma$ with vertices in $\Gamma \cap B$ so that the $\beta|y_0 - y_n|$ neighborhood of the vertices contain $\Gamma \cap B$, and so that the geodesic deviation (that is, its length minus $|y_0 - y_n|$ is at most $\beta|y_0 - y_n|$. In the last, we show that if $\beta'(x, r) < \beta$, there is $s : [0, 1] \rightarrow \ell^\infty$ whose geodesic deviation and whose distance from any point in $\Gamma \cap B$ are at most $\beta|s(0) - s(1)|$.

Observe that it does not make sense to study the length of a metric curve using the original β -number, even if we consider X as lying in some vector space. For example, if $X \subseteq L^1$ is the image of $s : [0, 1] \rightarrow L^1([0, 1])$ defined by $t \mapsto \mathbb{1}_{[0, t]}$, then this is a geodesic, so in particular, it is a rectifiable curve of finite length. However, $\beta(x, r)$ (i.e. the width of the smallest tube containing $X \cap B(x, r)$ in L^1 , rescaled by a factor r) is uniformly bounded away from zero. Even so, it is easy to check that $\hat{\beta}(x, r) = \beta'(x, r) = 0$ for all $x \in X$ and $r > 0$. This, of course, makes the terminology “wiggly” rather misleading in metric spaces, since there are certainly non-flat or highly “wiggly” geodesics in the general setting, which is why we restrict its use mainly to the title and previous work in order to

be consistent with the literature. Later on, however, we will show that in a Hilbert space we have for some $C > 0$,

$$\beta'(x, r) \leq \beta(x, r) \leq C\beta'(x, r)^{\frac{1}{2}}. \quad (1.8)$$

The fact that the two should be correlated in this setting seems natural as $\beta(x, r)$ is measuring how far Γ is deviating from a straight line, which are the only geodesics in Hilbert space.

In Lemma 18 below, we will also show that for some $C > 0$,

$$\beta'(x, r) \leq \hat{\beta}(x, r) \leq C\beta'(x, r)^{\frac{1}{2}}$$

so that Theorem 3 follows from the following theorem, which is the main result of this paper.

Theorem 4. *There is $c > 0$ such that the following holds. If Γ is a compact connected metric space and $\beta_{\Gamma}'(x, r) > \beta > 0$ for all $x \in \Gamma$ and $r \in (0, r_0)$ for some $r_0 > 0$, then $\dim \Gamma \geq 1 + c\beta^2$.*

We should warn the reader, however, that the quadratic dependence on β in this theorem and Theorem 1 appear for completely different reasons: in Theorem 1, it arises from using Theorem 2, or ultimately, from using the Pythagorean theorem, which is of course not present in the general metric space setting; in Theorem 4, it seems to be an artifact of the construction.

Our approach to proving Theorem 4 follows the original proof of Theorem 1 described above: to show that a metric curve Γ has large dimension, we approximate it by a polygonal curve, estimate its length from below and use this estimate to construct a Frostmann measure just like before, but in lieu of a traveling salesman theorem. (In fact, taking $\beta'(x, A2^{-n})$ instead of $\beta(x, A2^{-n})^2$ in Theorem 2 does not lead to a metric version of Theorem 2 for a similar reason that Hahlo-maa's β -number doesn't work; one need only adapt Schul's example [12, Section 3.3.1].)

The original context of Bishop and Tyson's conjecture concerned conformal dimension. In [2], it is shown that the antenna set has conformal dimension one yet no quasisymmetric image of it into any metric space has dimension equal to one. Recall that a *quasisymmetric map* $f : X \rightarrow Y$ between two metric spaces is a map for which there is an increasing homeomorphism $\eta : (0, \infty) \rightarrow (0, \infty)$ such that for any $x, y, z \in X$,

$$\frac{|f(x) - f(y)|}{|f(z) - f(y)|} \leq \eta \left(\frac{|x - y|}{|z - y|} \right).$$

The conformal dimension of a metric space X is

$$\text{C-dim} X = \inf_f \dim f(X)$$

where the infimum ranges over all quasimetric maps $f : X \rightarrow f(X)$.

The *antenna set* is a self similar fractal lying in \mathbb{C} whose similarities are the following:

$$f_1(z) = \frac{z}{2}, \quad f_2(z) = \frac{z+1}{2}, \quad f_3(z) = i\alpha z + \frac{1}{2}, \quad f_4(z) = -i\alpha z + \frac{1}{2} + i\alpha$$

where $\alpha \in (0, \frac{1}{2})$ is some fixed angle (see Figure 2).

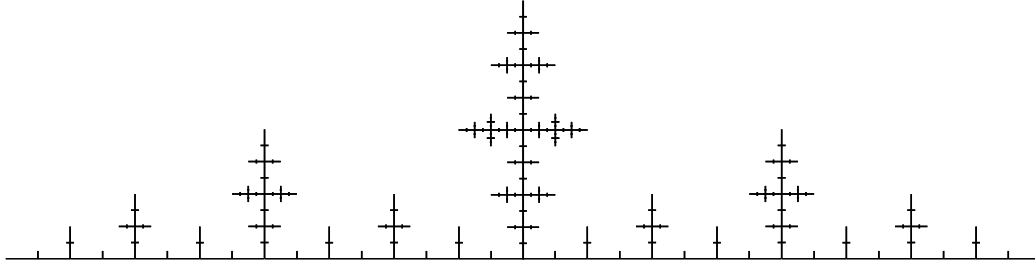


Figure 2: The antenna set with $\alpha = \frac{1}{4}$.

To show that the quasimetric image of this set is always larger than one, the authors show by hand that any quasimetric map of the antenna set naturally induces a Frostmann measure of dimension larger than one. At the end of the paper, however, the authors conjecture another way of showing this same result is by proving an analogue of Theorem 1 for a β -number which is uniformly large for the antenna set as well as any quasimetric image of it.

Theorem 4 doesn't just give a much longer proof of Bishop and Tyson's result, but it lends itself to more general sets lacking any self-similar structure.

Definition 5. Let $c > 0$, $Y = [0, e_1] \cup [0, e_2] \cup [0, e_3] \subseteq \mathbb{R}^3$, where e_j is the j th standard basis vector in \mathbb{R}^3 , and let X be a compact connected metric space in ℓ^∞ . For $x \in X$, $r > 0$, we say $B(x, r)$ has a c -antenna if there is a homeomorphism $h : Y \rightarrow h(Y) \subseteq B(x, r)$ such that

$$d(h(e_i), h([0, e_j] \cup [0, e_k])) \geq cr$$

for all permutations (i, j, k) of $(1, 2, 3)$. We say X is c -antenna-like if $B(x, r)$ has a c -antenna for every $x \in X$ and $r < \frac{\text{diam} X}{2}$,

Clearly, the classical antenna set in \mathbb{R}^2 is antenna-like.

Theorem 6. *Let X be a compact connected metric space in ℓ^∞ .*

1. *If $B(x, r)$ has a c -antenna, then $\beta'(x, r) > \frac{c}{7}$. Hence, if X is c -antenna-like, we have $\dim X \geq 1 + \frac{\kappa}{49}c^2$.*
2. *Any quasisymmetric image of an antenna-like set into any metric space is also antenna-like and hence has dimension strictly larger than one.*

In [14], Tyson and Wu show that the *quasiconformal dimension* of the Sierpinski Gasket $SG \subseteq \mathbb{R}^2$ (as well as other polygasket fractals) is one. This means that the infimum of $\dim f(SG)$ over all quasiconformal homeomorphisms $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is one. Theorem 6 shows in addition that this infimum is never attained even if one allows quasisymmetric images into arbitrary metric spaces.

Corollary 7. *If $K \subseteq \mathbb{R}^d$ is a path-wise connected self-similar fractal not contained in a line, then $\dim f(K) > 1$ for all $f : K \rightarrow f(K)$ quasisymmetric. In particular, $\dim f(SG) > 1$ for all quasisymmetric maps $f : SG \rightarrow f(SG)$.*

The proof is simple: if K satisfies the conditions of the theorem, then it must be c -antenna-like with some constant c .

At the end of this paper, we discuss the relation between the Euclidean and metric definitions of β discussed above. In [2], the authors mention that $\hat{\beta}_X(x, r)$ strictly bounded from zero if and only if the Euclidean β_X (defined in (1.1)) is also uniformly bounded away from zero in the case $X \subseteq \mathbb{R}^n$. This is certainly true, but the quantitative dependence between the two is not linear.

1.2 Outline

In Section 2, we go over some necessary notation and tools before proceeding to the main proof in Section 3. In Section 4, we prove Theorem 6, and in Section 5 we compare the β' , $\hat{\beta}$, and β .

1.3 Acknowledgements

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2 Preliminaries

2.1 Basic notation

Let X denote some metric space with metric d . By the Kuratowski embedding theorem, we may assume $X \subseteq \ell^\infty$ and that $d(x, y) = |x - y|$ for $x, y \in X$, where $|\cdot|$ is the norm on ℓ^∞ . For $x \in \ell^\infty$ and $r > 0$, we will write

$$B(x, r) = \{y \in \ell^\infty : |x - y| < r\} \subseteq \ell^\infty.$$

If $B = B(x, r)$ and $\lambda > 0$, we write λB for $B(x, \lambda r)$. For a set $A \subseteq \ell^\infty$, and $\delta > 0$, define

$$A_\delta = \{x \in \ell^\infty : d(x, A) < \delta\} \text{ and } \text{diam} A = \sup\{|x - y| : x, y \in A\}$$

where

$$d(A, B) = \inf\{|x - y| : x \in A, y \in B\}, \quad d(x, A) = d(\{x\}, A).$$

For a set $E \subseteq \mathbb{R}$, let $|E|$ denote its Lebesgue measure. For an interval $I \subseteq \mathbb{R}$, we will write a_I and b_I for its left and right endpoints respectively. For $s > 0$, $\delta \in (0, \infty]$ and $A \subseteq \ell^\infty$, define

$$\mathcal{H}_\delta^s(A) = \inf \left\{ \sum \text{diam} A_j : A \subseteq \bigcup A_j, \text{diam} A_j < \delta \right\},$$

$$\mathcal{H}^s(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A).$$

The *Hausdorff dimension* of a set A is

$$\dim A := \inf\{s : \mathcal{H}^s(A) = 0\}.$$

2.2 Cubes

In this section, we construct a family of subsets of ℓ^∞ , tailored to a metric space X , that have properties similar to dyadic cubes in Euclidean space. These cubes appeared in [13] (where they were alternatively called “cores”) and are variants of the so-called Christ-David Cubes, which originated in [4, 3].

Fix $M > 0$ and $c \in (0, \frac{1}{8})$. Let $X_n \subseteq X$ be a nested sequence of maximal M^{-n} -nets in X . Let

$$\mathcal{B}_n = \{B(x, M^{-n}) : x \in X_n\}, \quad \mathcal{B} = \bigcup_n \mathcal{B}_n.$$

For $B = B(x, M^{-n}) \in \mathcal{B}_n$, define

$$Q_B^0 = cB, \quad Q_B^j = Q_B^{j-1} \cup \bigcup \{cB : B \in \bigcup_{m \geq n} \mathcal{B}_m, cB \cap Q_B^{j-1} \neq \emptyset\}, \quad Q_B = \bigcup_{j=0}^{\infty} Q_B^j.$$

Basically, Q_B is the union of all balls B' that may be connected to B by a chain $\{cB_j\}$ with $B_j \in \mathcal{B}$, $\text{diam} B_j \leq \text{diam} B$, and $cB_j \cap cB_{j+1} \neq \emptyset$ for all j .

For such a cube Q constructed from $B(x, M^{-n})$, we let $x_Q = x$ and $B_Q = B(x, cM^{-n})$.

Let

$$\Delta_n = \{Q_B : B \in \mathcal{B}_n\}, \quad \Delta = \bigcup \Delta_n.$$

Observe that, for $Q \in \Delta_n$, $x_Q \in X_n$.

Lemma 8. *If $c < \frac{1}{8}$, then for X and Δ as above, the cubes Δ satisfy the following properties.*

1. *If $Q, R \in \Delta$ and $Q \cap R \neq \emptyset$, then $Q \subseteq R$ or $R \subseteq Q$.*

2. *For $Q \in \Delta$,*

$$B_Q \subseteq Q \subseteq (1 + 8M^{-1})B_Q. \quad (2.1)$$

The proof is essentially in [11], but with slightly different parameters. So that the reader need not perform the needed modifications, we provide a proof here.

Proof. The first part follows from the definition of the cubes Q . To prove the second part, we first claim that if $\{B_j\}_{j=0}^n$ is a chain of balls with centers x_j for which $cB_j \cap cB_{j+1} \neq \emptyset$, then for $C = \frac{1}{1-2M^{-1}}$,

$$\sum_{j=0}^n \text{diam} cB_j \leq C \max_{j=0, \dots, n} \text{diam} cB_j. \quad (2.2)$$

We prove by induction. Let x_j denote the center of B_j . If $n = 1$, $\text{diam} B_0 \leq \text{diam} B_1$, and x_0 and x_1 are the centers of B_0 and B_1 respectively, then $\text{diam} B_0 \leq M^{-1} \text{diam} B_1$ since otherwise $B_0, B_1 \in \mathcal{B}_N$ for some N and

$$M^{-n} \leq |x_0 - x_1| \leq 2cM^{-n} < M^{-n}$$

since $c < \frac{1}{8}$, which is a contradiction. Hence,

$$\text{diam} cB_0 + \text{diam} cB_1 \leq (1 + 2M^{-1}) \text{diam} cB_1 \leq C \text{diam} cB_1.$$

Now suppose $n > 1$. Let $j_0 \in \{1, \dots, n\}$ be so that

$$\text{diam} B_{j_0} = \max \text{diam} B_j = 2M^{-N}. \quad (2.3)$$

Let $i_0 \leq j_0$ be the minimal integer for which $\text{diam} B_{i_0} \leq M^{-1} \text{diam} B_{j_0}$, and let $k_0 \geq j_0$ be the maximal integer such that $B_{k_0} \leq M^{-1} B_{j_0}$. Then by the induction hypothesis,

$$\sum_{j=j_0+1}^{k_0} \text{diam} cB_j \leq C \max_{j_0 < j \leq k_0} \text{diam} cB_j \leq CM^{-1} \text{diam} cB_{j_0}$$

and

$$\sum_{j=i_0}^{j_0-1} \text{diam} cB_j \leq C \max_{i_0 \leq j < j_0} \text{diam} cB_j \leq CM^{-1} \text{diam} cB_{j_0} \quad (2.4)$$

so that

$$\sum_{j=i_0}^{k_0} \text{diam} B_j \leq (1 + 2CM^{-1}) \text{diam} cB_{j_0} = C \text{diam} cB_{j_0}. \quad (2.5)$$

Observe that if $i_0 > 0$, then

$$\begin{aligned} |x_{i_0-1} - x_{j_0}| &\leq \sum_{i=i_0-1}^{j_0} \text{diam} cB_i \leq \text{diam} cB_{i_0-1} + \text{diam} cB_{j_0} + \sum_{i=i_0}^{j_0-1} 2cB_{j_0} \\ &\stackrel{(2.3), (2.4)}{\leq} 2\text{diam} cB_{j_0} + CM^{-1} \text{diam} cB_{j_0} \\ &\leq (2c + cCM^{-1}) \text{diam} B_{j_0} = (2c + cCM^{-1}) 2M^{-N} \\ &< M^{-N} \end{aligned}$$

for $c < \frac{1}{4}$ and $M > 4$ (note that this makes $C < 2$). Thus, $x_{i_0-1} \notin X_N$ and $B_{i_0-1} \notin \mathcal{B}_N$, so

$$\text{diam} B_{i_0-1} \leq M^{-1} \text{diam} B_{j_0},$$

which contradicts the minimality of i_0 , hence $i_0 = 0$. We can prove similarly that $k_0 = n$, and this with (2.4) proves (2.2). This in turn implies that for $Q \in \Delta_N$,

$$\begin{aligned} Q &\subseteq B(x_Q, cM^{-N} + (C - 1)\text{diam} cB_j) \\ &= B(x_Q, c(1 + \frac{4M^{-1}}{1 - 2M^{-1}})M^{-N}) \\ &\subseteq (1 + 8M^{-1})B_Q. \end{aligned}$$

□

For N large enough, this means we can pick our cubes so that they don't differ much from balls. We will set $8M^{-1} = \varepsilon\beta$ for some $\varepsilon \in (0, 1)$ to be determined later, so that

$$B_Q \subseteq Q \subseteq (1 + \varepsilon\beta)B_Q \quad (2.6)$$

Remark 9. For those familiar with Christ-David cubes, they will observe that these cubes we construct here are of course very different. In particular, each Δ_n does not partition our metric space (in the same way that dyadic cubes of side length 2^{-n} would partition Euclidean space, for example). However, for each n we do have

$$X \subseteq \bigcup_{Q \in \Delta_n} \frac{1}{c}Q, \quad (2.7)$$

and we still have the familiar intersection properties in Lemma 8. The reason for the somewhat ad hoc construction is the crucial “roundness” property (2.6).

Lemma 10. *Let $\Gamma = \gamma([0, 1])$, where γ is arclength parametrized, i.e.*

$$\ell(\gamma|_{[a,b]}) = b - a \text{ for all } [a, b] \subseteq [0, 1]$$

and let Δ be the cubes from Lemma 8. Then for any $Q \in \Delta$, $\mathcal{H}^1(\partial Q) = 0$.

Proof. Firstly, recall that for \mathcal{H}^1 -a.e. $z \in \Gamma$,

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^1(B(x, r) \cap \Gamma)}{2r} = 1. \quad (2.8)$$

Indeed, it is easy to show $\limsup_{r \rightarrow 0} \frac{\mathcal{H}^1(B(x, r) \cap \Gamma)}{2r} \leq 1$ for a.e. $z \in \Gamma$ by standard geometric measure theory. Furthermore, by [8, Theorem 2] and the fact that γ is an arc length parametrization,

$$\sup_{w, t \in B(s, \rho)} ||\gamma(w) - \gamma(t)| - |w - t|| = o(\rho) \quad (2.9)$$

for almost every $s \in [0, 1]$. In particular, for such an s , and for every $\varepsilon > 0$,

$$d(\gamma([s - r, s - \varepsilon r]), \gamma([s + \varepsilon r, s + r])) > 0 \text{ as } r \rightarrow 0$$

and thus they are disjoint for small values of r . Hence, since γ is 1-Lipschitz,

$$\begin{aligned}
\liminf_{r \rightarrow 0} \frac{\mathcal{H}^1(B(\gamma(s), r))}{2r} &\geq \liminf_{r \rightarrow 0} \frac{\mathcal{H}^1(\gamma(B(s, r)))}{2r} \\
&\geq \liminf_{r \rightarrow 0} \frac{\mathcal{H}^1(\gamma([s-r, s-\varepsilon r])) + \mathcal{H}^1(\gamma([s+\varepsilon r, s+r]))}{2r} \\
&\geq \liminf_{r \rightarrow 0} \frac{|\gamma(s-r) - \gamma(s-\varepsilon r)| + |\gamma(s+\varepsilon r) - \gamma(s+r)|}{2r} \\
&\stackrel{(2.9)}{=} \frac{(1-\varepsilon)r + (1-\varepsilon)r}{2r} = 1 - \varepsilon
\end{aligned}$$

and letting $\varepsilon \rightarrow 0$ gives (2.8).

Now, let $Q \in \Delta_n$, and observe that for each $z \in \partial Q$, there is $x_j \in X_j$ with

$$|z - x_j| \leq 2M^{-j}. \quad (2.10)$$

Then for $j > n$ either $Q_{B(x_j, cM^{-j})} \subseteq Q$ or $Q_{B(x_j, cM^{-j})} \cap Q = \emptyset$ (recall that these sets are open and don't intersect at their boundaries). In either case,

$$B(x_j, \frac{c}{2}M^{-j}) \subseteq B(x_j, cM^{-j}) \subseteq Q_{B(x_j, cM^{-j})} \subseteq (\partial Q)^c$$

so that

$$B(x_j, \frac{c}{2}M^{-j}) \cap \partial Q = \emptyset.$$

Moreover,

$$B(x_j, \frac{c}{2}M^{-j}) \stackrel{(2.10)}{\subseteq} B(z, (\frac{c}{2} + 2)M^{-j}) \subseteq B(z, 4M^{-j}).$$

Hence, for \mathcal{H}^1 -a.e. $z \in \partial Q$,

$$\begin{aligned}
\limsup_{j \rightarrow 0} \frac{\mathcal{H}^1(\partial Q \cap B(z, 4M^{-j}))}{8M^{-j}} \\
&\leq \limsup_{j \rightarrow 0} \frac{\mathcal{H}^1(B(z, 4M^{-j})) - \mathcal{H}^1(B(x_j, \frac{c}{2}M^{-j}))}{8M^{-n}} \\
&\leq \limsup_{j \rightarrow 0} \frac{\mathcal{H}^1(B(z, 4M^{-j})) - \frac{c}{2}M^{-j}}{8M^{-n}} \stackrel{(2.8)}{\leq} 1 - \frac{c}{16} < 1,
\end{aligned}$$

where in the penultimate inequality we used the fact that Γ was connected to imply that $\mathcal{H}^1(B(x_j, \frac{c}{2}M^{-j})) \geq \frac{c}{2}M^{-j}$. By (2.8), we must have $\mathcal{H}^1(\partial Q) = 0$.

Finally, by [8, Lemma 4], since γ is arc length parametrized, we may partition almost all of $[0, 1]$ into Borel sets E_i upon each of which γ is C_i -bi-Lipschitz for some $C_i > 1$, hence

$$|\gamma^{-1}(\partial Q)| \leq \sum_i |\gamma|_{E_i}^{-1}(\partial Q)| \leq \sum_i C_i \mathcal{H}^1(\gamma(E_i) \cap \partial Q) \leq \sum_i C_i \mathcal{H}^1(\partial Q) = 0.$$

□

The following lemma will be used frequently.

Lemma 11. *Let $I \subseteq \mathbb{R}$ be an interval, $s : I \rightarrow \ell^\infty$ be continuous and $I' \subseteq I$ a subinterval. Then*

$$\ell(s|_{I'}) - |s(a_{I'}) - s(b_{I'})| \leq \ell(s|_I) - |s(a_I) - s(b_I)|. \quad (2.11)$$

Proof. We may assume $\ell(s_I) < \infty$, otherwise there is nothing to show. We simply estimate

$$\begin{aligned} \ell(s|_{I'}) - |s(a_{I'}) - s(b_{I'})| &= \ell(s|_I) - \ell(s|_{I \setminus I'}) - |s(a_{I'}) - s(b_{I'})| \\ &\leq \ell(s|_I) - (|s(a_I) - s(a_{I'})| + |s(b_I) - s(b_{I'})|) - |s(a_{I'}) - s(b_{I'})| \\ &\leq \ell(s|_I) - |s(a_I) - s(b_I)|. \end{aligned}$$

□

3 Proof of Theorem 4

3.1 Setup

For this section, we fix a compact connected set Γ satisfying the conditions of Theorem 4. The main tool is the following Lemma, which can be seen as a very weak substitute for Theorem 2.

Lemma 12. *Let $c' < \frac{1}{8}$. We can pick M large enough (by picking $\varepsilon > 0$ small enough) and pick $\kappa > 0$ such that, for any Γ satisfy the conditions of Theorem 4 for some $\beta > 0$, the following holds. If X_n is any nested sequence of M^{-n} -nets in Γ , there is $n_0 = n_0(\beta)$ such that for $x_0 \in X_n$ with $M^{-n} < \min\{r_0, \frac{\text{diam} X}{2}\}$,*

$$\#X_{n+n_0} \cap B(x_0, c'M^{-n}) \geq M^{(1+\kappa\beta^2)n_0}. \quad (3.1)$$

Let us first explain why this proves Theorem 4.

Proof of Theorem 4. Let Δ be the cubes from Lemma 8 tailored to the metric space Γ with $c = c'$ and define inductively,

$$\Delta'_0 = \Delta_0$$

$$\Delta'_{n+1} = \{R \in \Delta_{(n+1)n_0} : R \subseteq Q \text{ for some } Q \in \Delta'_n\}.$$

By Lemma 12, for any $Q \in \Delta'_n$, if $B_Q = B(x_Q, cM^{-N})$, then

$$\#\{R \in \Delta'_{n+1}, R \subseteq Q\} \geq \#X_{N+n_0} \cap Q \geq \#X_{n_0} \cap c'B_Q \geq M^{(1+\kappa\beta^2)n_0} \quad (3.2)$$

and moreover, since $c' < \frac{1}{8}$,

$$2B_Q \cap 2B_R = \emptyset \text{ for } Q, R \in \Delta_n. \quad (3.3)$$

Define a probability measure μ inductively by picking $Q_0 \in \Delta'_0$, setting $\mu(Q_0) = 1$ and for $Q \in \Delta'_n$ and $R \in \Delta'_{n+1}$, $R \subseteq Q$

$$\frac{\mu(R)}{\mu(Q)} = \frac{1}{\#\{S \in \Delta'_{n+1} : S \subseteq Q\}} \stackrel{(3.2)}{\leq} M^{-(1+\kappa\beta^2)n_0}. \quad (3.4)$$

Let $x \in \Gamma$, $r \in (0, \frac{r_0}{M})$. Pick n so that

$$M^{-n_0(n+1)} \leq r < M^{-n_0n}. \quad (3.5)$$

Claim: There is at most one $y \in X_{(n-1)n_0}$ such that

$$B(y, c'M^{-(n-1)n_0}) \cap B(x, r) \neq \emptyset \text{ and } Q = Q_{B(y, c'M^{-(n-1)n_0})} \in \Delta'_{n-1}. \quad (3.6)$$

Indeed, if there were another such $y' \in X_{(n-1)n_0}$ with $B(y', c'M^{-(n-1)n_0}) \cap B(x, r) \neq \emptyset$, then

$$\begin{aligned} M^{-(n-1)n_0} &\leq |y' - y| \\ &\leq c'M^{-(n-1)n_0} + d(B(y, c'M^{-(n-1)n_0}), B(y', c'M^{-(n-1)n_0})) + c'M^{-(n-1)n_0} \\ &\leq 2c'M^{-(n-1)n_0} + \text{diam}B(x, r) \\ &\stackrel{(3.5)}{\leq} 2c'M^{-(n-1)n_0}(c' + M^{-n_0}) \\ &< 4c'M^{-(n-1)n_0} < M^{-(n-1)n_0} \end{aligned}$$

since $c' < \frac{1}{8}$ and we can pick $\varepsilon < \frac{c'}{8}$ so that $M^{-n_0} \leq M^{-1} < c'$, which gives a contradiction and proves the claim.

Now, assuming we have $y \in X_{(n-1)n_0}$ satisfying (3.6),

$$\begin{aligned} B(x, r) &\subseteq B(y, c'M^{-(n-1)n_0} + 2r) \\ &\subseteq B(y, c'M^{-(n-1)n_0} + 2M^{-nn_0}) \\ &\subseteq B(y, 2c'M^{-(n-1)n_0}) \\ &= 2B_Q \end{aligned}$$

for M large enough (that is, for $2M^{-1} < c'$, which is possibly by picking $\varepsilon < \frac{c'}{16}$). If $Q \notin \Delta'_{n-1}$, then (3.3) implies $2B_Q \cap 2B_R = \emptyset$ for all $R \in \Delta'_{n-1}$, and so

$$\mu(B(x, r)) \leq \mu(2B_Q) = 0.$$

Otherwise, if $Q \in \Delta'_{n-1}$, then $Q \subseteq Q_0$, so that

$$\mu(B(x, r)) \leq \mu(2B_Q) \stackrel{(3.3)}{=} \mu(Q) \stackrel{(3.4)}{=} M^{-(1+\kappa\beta^2)n_0(n-1)} \mu(Q_0) \lesssim_{M,\beta,\kappa} r^{-(1+\kappa\beta^2)}$$

which implies $\dim \Gamma \geq 1 + \kappa\beta^2$. □

To show (3.1), we will approximate Γ by a tree containing a sufficiently dense net in Γ and estimate its length from below. The following lemma relates the length of this tree to the number of net points in Γ .

Lemma 13. *Let \tilde{X}_{n_0} be a maximal M^{-n_0} -net for a connected metric space Γ where n_0 is so that $4M^{-n_0} < \frac{\text{diam} X}{4}$. Then we may embed Γ into ℓ^∞ so that there is a connected set $\Gamma_{n_0} \subseteq \ell^\infty$ containing \tilde{X}_{n_0} such that for any $x \in \tilde{X}_{n_0}$ and $r \in (4M^{-n_0}, \frac{\text{diam} X}{4})$,*

$$\mathcal{H}^1(\Gamma_{n_0} \cap B(x, \frac{r}{2})) \leq 4M^{-n_0} \#(\tilde{X}_{n_0} \cap B(x, r)). \quad (3.7)$$

Proof. Embed Γ into $\ell^\infty(\mathbb{N})$ so that for any $x \in \Gamma$, the first $\#\tilde{X}_{n_0}$ coordinates are all zero. Construct a sequence of trees T_j as follows. Enumerate the elements of $X_{n_0} = \{x_1, \dots, x_{\#\tilde{X}_{n_0}}\}$. For two points xy , let

$$A_{xy,i} = \{tx + (1-t)y + \max\{t, 1-t\}|x-y|e_i : t \in [0, 1]\}$$

where e_i is the standard basis vector in $\ell^\infty(\mathbb{N})$ (i.e. it is equal to 1 in the i th coordinate and zero in every other coordinate).

Now construct a sequence of trees T_j in $\ell^\infty \oplus \mathbb{R}^{\#\tilde{X}_{n_0}} \cong \ell^\infty$ inductively by setting $T_0 = \{x_0\}$ and T_{j+1} equal to T_j united with $S_{j+1} := A_{x_{j+1}x'_{j+1},j+1}$, where $x'_{j+1} \in \{x_1, \dots, x_j\}$ and $x_{j+1} \in \tilde{X}_{n_0} \setminus \{x_1, \dots, x_j\}$ are such that

$$|x_{j+1} - x'_{j+1}| = d(\tilde{X}_{n_0} \setminus \{x_1, \dots, x_j\}, \{x_1, \dots, x_j\}).$$

Since Γ is connected, $|x_{j+1} - x'_{j+1}| \leq 2M^{-n_0}$, so that

$$\mathcal{H}^1(S_j) = \mathcal{H}^1(A_{x_j, x'_j, j}) \leq 2|x_j - x'_j| \leq 4 \cdot 2M^{-n_0}.$$

Then $\Gamma_{n_0} := T_{\#\tilde{X}_{n_0}}$ is a tree contained in $\ell^\infty \oplus \mathbb{R}^{\#\tilde{X}_{n_0}}$ containing \tilde{X}_{n_0} (the reason we made the arcs S_{j+1} reach into an alternate dimension is to guarantee that the branches of the tree don't intersect except at the points \tilde{X}_{n_0}).

To show (3.7), observe that since $\frac{r}{2} > 2M^{-n_0}$ and

$$x_j \in S_j \subseteq B(x_j, 2M^{-n_0}),$$

we have

$$\begin{aligned} \mathcal{H}^1(\Gamma_{n_0} \cap B(x, \frac{r}{2})) &\leq \sum_{S_j \cap B(x, \frac{r}{2}) \neq \emptyset} \mathcal{H}^1(S_j) \leq \sum_{x_j \in B(x, \frac{r}{2} + 2M^{-n_0})} 4M^{-n_0} \\ &\leq 4\#(\tilde{X}_{n_0} \cap B(x, r)). \end{aligned}$$

□

3.2 Proof of Lemma 12

We now dedicate ourselves to the proof of Lemma 12. Again, let Γ be a connected metric space satisfying the conditions of Theorem 4. Without loss of generality, $n = 0$. Embed Γ into ℓ^∞ as in Lemma 13. Fix $x_0 \in X_0$ and $n_0 \in \mathbb{N}$. Let Γ_{n_0} be the tree from Lemma 13 containing the M^{-n_0} -net \tilde{X} in Γ . Assume $\text{diam}\Gamma > 2$ so that $\text{diam}\Gamma_{n_0} > 1$.

Since Γ_{n_0} is a tree of finite length, it is not hard to show that there is an arc length parametrized path $\gamma : [0, 2\mathcal{H}^1(\Gamma_{n_0})] \rightarrow \Gamma_{n_0}$ that traverses almost every point in Γ_{n_0} at most twice (except at the discrete set of points \tilde{X}). To see this,

we first prove the well known statement for graphs, that is, if T is a tree with finitely many vertices V and edges E , then for any vertex $x \in V$ there is a path that traverses every edge exactly twice and begins and ends at x . If $\#V = 2$, then there is only one edge, and the path is just the one that travels along the edge once and back again from x . Inductively, if $\#V = j + 1$, pick a vertex $x \in V$ that disconnects V into at least two components. Each component is a tree with at most j vertices, and by the induction hypothesis, there is a path traversing each component, and every edge in that component exactly twice, that starts and ends at x . We concatenate each of these paths into one whole one, and this proves the claim. A similar construction builds an arc length path $\gamma : [0, 2\mathcal{H}^1(\Gamma_{n_0})] \rightarrow \Gamma_{n_0}$ with the desired properties.

Let Δ be the cubes from Lemma 8 for $X = \Gamma_{n_0}$ and fix $Q_0 \in \Delta_0$ so that $x_{Q_0} = x$. We will adjust the values of $c > 0$ in Lemma 8 and the value $\varepsilon > 0$ in the definition of M as we go along the proof. For $Q, R \in \Delta$, write $R^1 = Q$ if R is a maximal cube in Δ properly contained in Q . For $n \geq 0$ and $Q \in \Delta$, define

$$\mathcal{L}_1(Q) = \{R \in \Delta : R^1 = Q\}$$

$$\mathcal{L}_n(Q) = \bigcup_{R \in \mathcal{L}_{n-1}(Q)} \mathcal{L}_1(R),$$

$$\tilde{\mathcal{L}}_n(Q) = \mathcal{L}_n(Q) \cap \bigcup_{j=0}^{n_0-1} \Delta_j.$$

$$\tilde{\mathcal{L}}(Q) = \bigcup \tilde{\mathcal{L}}_n(Q)$$

$$\tilde{\mathcal{L}}_n = \tilde{\mathcal{L}}_n(Q_0), \quad \tilde{\mathcal{L}} = \tilde{\mathcal{L}}(Q_0).$$

For $Q \in \Delta$, let

$$\lambda(Q) = \{[a, b] : (a, b) \text{ a connected component of } \gamma^{-1}(Q)\}$$

and for $n \leq n_0$, define γ_n to be the continuous function such that for all $Q \in \mathcal{L}_n(Q_0)$ and $[a, b] \in \lambda(Q)$,

$$\gamma_n|_{[a,b]}(at + (1-t)b) = t\gamma(a) + (1-t)\gamma(b) \text{ for } t \in [0, 1],$$

that is, γ_n is linear in all cubes in Δ_n and agrees with γ on the boundaries of the cubes (see Figure 3).

Lemma 12 will follow from the following two lemmas:

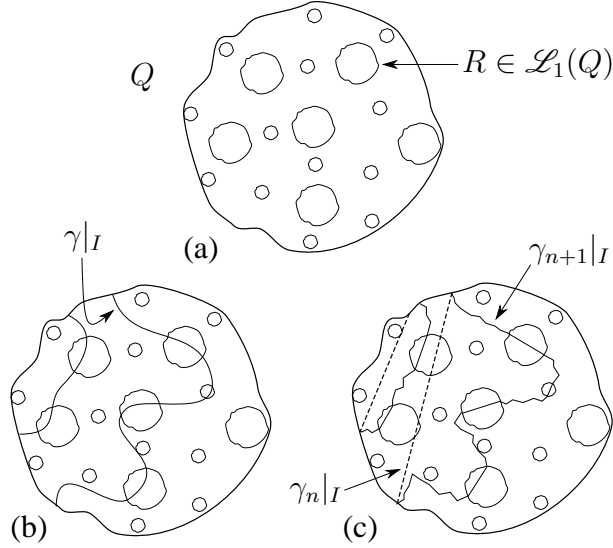


Figure 3: In (a), we have a typical cube $Q \in \Delta_n$, and some of its children in $\mathcal{L}_1(Q)$. Observe that their sizes can be radically different. In (b) are the components $\gamma|_{\gamma^{-1}(Q)}$, where in this case $\gamma^{-1}(Q)$ consists of two intervals, and we've pointed at a particular component $\gamma|_I$ for some $I \in \lambda(Q)$. In (c), the dotted lines represent the components of $\gamma_n|_{\gamma^{-1}(Q)}$, which is affine in cubes in Δ_n , and hence is affine in Q , and the components of $\gamma_{n+1}|_{\gamma^{-1}(Q)}$, which are affine in the children of Q (since they are in Δ_{n+1}).

Lemma 14. *There is $K \in (0, 1)$ and $\beta_0 > 0$ (independent of n_0 above) such that if $\beta \in (0, \beta_0)$, $n < n_0$, and $Q \in \tilde{\mathcal{L}}_n$, either*

$$\sum_{I \in \lambda(Q)} (\ell(\gamma_{n+1}|_I) - \ell(\gamma_n|_I)) \geq \frac{\varepsilon\beta}{4} \text{diam} Q \quad (3.8)$$

or $Q \in \Delta_{Bad}$, where

$$\Delta_{Bad} = \{R \in \tilde{\mathcal{L}} : \mathcal{H}_\infty^1(\Gamma_{n_0} \cap R) \geq (1 + K\beta) \text{diam} R\} \quad (3.9)$$

Lemma 15. *With Δ_{Bad} defined as above, we have*

$$\sum_{Q \in \Delta_{Bad}} \beta \text{diam} Q \leq \frac{2}{K\varepsilon} \mathcal{H}^1(\Gamma_{n_0}).$$

Indeed, for $Q \in \tilde{\mathcal{L}}$, let $n(Q)$ be such that $Q \in \mathcal{L}_n$ and define

$$d(Q) = \sum_{I \in \lambda(Q)} (\ell(\gamma_{n(Q)+1}|_I) - \ell(\gamma_{n(Q)}|_I)).$$

By telescoping sums, and Lemma 10, we have

$$\begin{aligned} \sum_{Q \in \tilde{\mathcal{L}}} d(Q) &= \sum_{n=0}^{n_0-1} \sum_{Q \in \tilde{\mathcal{L}}_n} \sum_{I \in \lambda(Q)} (\ell(\gamma_{n+1}|_I) - \ell(\gamma_n|_I)) \\ &= \sum_{n=0}^{n_0-1} (\ell(\gamma_{n+1}|_{\gamma^{-1}(Q_0)}) - \ell(\gamma_n|_{\gamma^{-1}(Q_0)})) \\ &\leq \ell(\gamma|_{\gamma^{-1}(Q_0)}) = 2\mathcal{H}^1(\Gamma_{n_0} \cap Q_0). \end{aligned} \quad (3.10)$$

Observe that $\text{diam}(\Gamma_{n_0} \cap Q_0) \geq 1$ since $Q_0 \in \Delta_0$, $\text{diam}\Gamma_{n_0} > 1$, and Γ_{n_0} is connected. This, Lemma 14, and Lemma 15 imply

$$\begin{aligned} \frac{10}{K\varepsilon} \mathcal{H}^1(\Gamma_{n_0} \cap Q_0) &\geq \frac{2}{K\varepsilon} \mathcal{H}^1(\Gamma_{n_0} \cap Q_0) + \frac{8}{\varepsilon} \mathcal{H}^1(\Gamma_{n_0} \cap Q_0) \\ &\geq \sum_{Q \in \Delta_{Bad}} \beta \text{diam} Q + \frac{4}{\varepsilon} \sum_{Q \in \tilde{\mathcal{L}} \setminus \Delta_{Bad}} d(Q) \\ &\geq \sum_{Q \in \Delta_{Bad}} \beta \text{diam} Q + \sum_{Q \in \tilde{\mathcal{L}} \setminus \Delta_{Bad}} \beta \text{diam} Q \\ &= \sum_{n=0}^{n_0-1} \sum_{Q \in \Delta_n} \beta \text{diam} Q \geq \sum_{n=0}^{n_0-1} \sum_{Q \in \Delta_n} \beta \text{diam} B_Q \\ &= \sum_{n=0}^{n_0-1} c \sum_{Q \in \Delta_n} \beta \text{diam} \frac{1}{c} B_Q \\ &\stackrel{(2.7)}{\geq} cn_0 \beta \text{diam}(\Gamma_{n_0} \cap Q_0) \geq cn_0 \beta \end{aligned}$$

so that

$$\frac{Kcn_0\beta\varepsilon}{10} \leq \mathcal{H}^1(\Gamma_{n_0} \cap Q_0)$$

and by Lemma 13

$$\begin{aligned} \mathcal{H}^1(\Gamma_{n_0} \cap Q_0) &\leq \mathcal{H}^1(\Gamma_{n_0} \cap (1 + \varepsilon\beta)B_{Q_0}) \leq \mathcal{H}^1(\Gamma_{n_0} \cap B(x, 2c)) \\ &\leq 4\#(\tilde{X}_{n_0} \cap B(x, 4c))M^{-n_0} \end{aligned}$$

Combining these two estimates we have, for $c < \frac{c'}{4}$ that

$$\delta n_0 M^{n_0} \beta \text{diam} \Gamma_{n_0} \leq \#(\tilde{X}_{n_0} \cap B(x_0, c')), \quad \delta = \frac{K c \varepsilon}{40}$$

Pick $n_0 = \left\lceil \frac{8}{\delta \beta^2 \varepsilon} \right\rceil$. Since $\frac{8}{\varepsilon \beta} = M$, we get

$$\begin{aligned} \#(\tilde{X}_{n_0} \cap B(x_0, c')) &\geq \delta n_0 M^{n_0} \beta = n_0 \left(\frac{\delta \varepsilon \beta^2}{8} \right) M^{n_0} \frac{8}{\varepsilon \beta} \geq M^{n_0+1} \\ &= M^{n_0(1+\frac{1}{n_0})} \geq M^{n_0(1+\frac{1}{\frac{8}{\delta \beta^2 \varepsilon}-1})} \geq M^{n_0(1+\frac{\delta}{16}\beta^2)} \end{aligned}$$

since $\frac{8}{\delta \beta^2} \geq 2$, and this proves Lemma 12 with $\kappa = \frac{\delta}{16}$.

Remark 16. By inspecting the proof of Lemma 14 below, one can solve for explicit values of ε, c, β_0 , and K . In particular, one can choose $\varepsilon < \frac{1}{12288}$, $K < \frac{1}{4096}$, $c < \frac{1}{64}$, and $\beta_0 = \frac{1}{356}$, so that the supremum of permissible values of κ is at least $(30 \cdot 2^{36})^{-1}$. This is by no means tight and can surely be improved.

In the next two subsections, we prove Lemma 14 and Lemma 15.

3.3 Proof of Lemma 14

First observe that for any $I \in \lambda(Q)$,

$$\begin{aligned} \ell(\gamma_{n+1}|_I) - \ell(\gamma_n|_I) &\geq \ell(\gamma_{n+1}|_I) - |\gamma_n(a_I) - \gamma_n(b_I)| \\ &= \ell(\gamma_{n+1}|_I) - |\gamma_{n+1}(a_I) - \gamma_{n+1}(b_I)| \geq 0. \end{aligned}$$

Hence it suffices to find just one interval $I \in \lambda(Q)$ for which

$$\ell(\gamma_{n+1}|_I) - \ell(\gamma_n|_I) \gtrsim \beta \text{diam} Q$$

Fix N so that $Q \in \Delta_N$. Let $\tilde{Q} \in \Delta_{N+1}$ be such that

$$x_Q \in \tilde{Q} \subset \tilde{Q}^1 = Q$$

and pick $I \in \lambda(Q)$ such that $\gamma_{n+1}(I) \cap \tilde{Q} \neq \emptyset$. Then since $\gamma_{n+1}|_I$ is a path entering Q , hitting \tilde{Q} , and then leaving Q , we can estimate

$$\begin{aligned}
\ell(\gamma_{n+1}|_I) &\geq 2d(\tilde{Q}, Q) \stackrel{(2.6)}{\geq} 2d((1 + \varepsilon\beta)B_{\tilde{Q}}, B_Q) \\
&= 2(cM^{-N} - (1 + \varepsilon\beta)cM^{-N-1}) \\
&= 2cM^{-N}(1 - (1 + \varepsilon\beta)M^{-1}) \\
&\geq \text{diam}B_Q(1 - \frac{\varepsilon\beta}{8} - \frac{\varepsilon^2\beta^2}{8}) \\
&> (1 - \varepsilon\beta)\text{diam}Q \\
&\geq \frac{1 - \varepsilon\beta}{1 + \varepsilon\beta}\text{diam}Q \\
&= \left(\frac{1 + \varepsilon\beta}{1 + \varepsilon\beta} - \frac{2\varepsilon\beta}{1 + \varepsilon\beta} \right) \text{diam}Q \\
&\geq (1 - 2\varepsilon\beta)\text{diam}Q.
\end{aligned} \tag{3.11}$$

Also observe that $\gamma_n|_I \subseteq Q$ is a line segment with endpoints the same as $\gamma_{n+1}|_I$, so that

$$\begin{aligned}
\ell(\gamma_n|_I) &= \mathcal{H}^1(\gamma_n(I)) = \text{diam}\gamma_n(I) = |\gamma_n(a_I) - \gamma_n(b_I)| \\
&= |\gamma_{n+1}(a_I) - \gamma_{n+1}(b_I)|
\end{aligned} \tag{3.12}$$

We consider a few cases.

Case 1: Suppose $\ell(\gamma_n(I)) < \frac{\text{diam}Q}{4}$. Recall that since γ_n is constant on I ,

$$\ell(\gamma_{n+1}|_I) - \ell(\gamma_n|_I) \stackrel{(3.11)}{\geq} (1 - 2\varepsilon\beta)\text{diam}Q - \frac{\text{diam}Q}{4} \geq \frac{\text{diam}Q}{8}$$

if $\varepsilon < \frac{1}{16}$, which implies the lemma in this case.

Case 2: Suppose

$$\ell(\gamma_n(I)) \geq \frac{\text{diam}Q}{4} > \frac{\varepsilon\beta}{4}\text{diam}Q. \tag{3.13}$$

We again split into two cases.

Case 2a: Suppose

$$\ell(\gamma_{n+1}|_I) \geq (1 + \varepsilon\beta)\ell(\gamma_n|_I).$$

Then clearly,

$$\ell(\gamma_{n+1}|_I) - \ell(\gamma_n|_I) \geq \varepsilon\beta\ell(\gamma_n|_I) \stackrel{(3.13)}{\geq} \frac{\varepsilon\beta}{4}\text{diam}Q.$$

Case 2b: Now suppose

$$\ell(\gamma_{n+1}|_I) < (1 + \varepsilon\beta)\ell(\gamma_n(I)). \quad (3.14)$$

Note that in this case, we have a better lower bound on $\ell(\gamma_n|_I)$, namely,

$$\ell(\gamma_n(I)) \stackrel{(3.14)}{\geq} \frac{\ell(\gamma_{n+1}|_I)}{1 + \varepsilon\beta} \stackrel{(3.11)}{\geq} \frac{1 - 2\varepsilon\beta}{1 + \varepsilon\beta}\text{diam}Q \geq (1 - 3\varepsilon\beta)\text{diam}Q. \quad (3.15)$$

Let $C \in (0, 1)$ (we will pick it's value later).

Lemma 17. *Let I' be the smallest interval in I so that*

$$\gamma_{n+1}(a_{I'}), \gamma_{n+1}(b_{I'}) \in \partial((1 - C\varepsilon\beta)B_Q)$$

and $\gamma_{n+1}(I') \cap \tilde{Q} \neq \emptyset$. Then

$$\ell(\gamma_{n+1}|_{I'}) - |\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| \leq 2\varepsilon\beta|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| \quad (3.16)$$

Proof. We first compute

$$\ell(\gamma_{n+1}|_{I'}) \geq 2d(\tilde{Q}, (1 - C\varepsilon\beta)B_Q) \quad (3.17)$$

$$\begin{aligned} &\stackrel{(2.6)}{\geq} 2d((1 + \varepsilon\beta)B_{\tilde{Q}}, (1 - C\varepsilon\beta)B_Q) \\ &= 2((1 - C\varepsilon\beta)cM^{-N} - (1 + \varepsilon\beta)cM^{-N-1}) \\ &= 2cM^{-N}(1 - C\varepsilon\beta - (1 + \varepsilon\beta)M^{-1}) \\ &\geq \text{diam}B_Q(1 - C\varepsilon\beta - 2M^{-1}) \\ &= (1 - C\varepsilon\beta - \frac{\varepsilon\beta}{4})\text{diam}B_Q \\ &\stackrel{(2.6)}{\geq} \frac{1 - C\varepsilon\beta - \frac{\varepsilon\beta}{4}}{1 + \varepsilon\beta}\text{diam}Q \\ &= \left(\frac{1 + \varepsilon\beta}{1 + \varepsilon\beta} - \frac{C\varepsilon\beta - \frac{5\varepsilon\beta}{4}}{1 + \varepsilon\beta} \right) \text{diam}Q \\ &\geq (1 - C\varepsilon\beta - 2\varepsilon\beta)\text{diam}Q \end{aligned} \quad (3.18)$$

Hence,

$$\begin{aligned}
& |\gamma_{n+1}(a_I) - \gamma_{n+1}(b_I)| - |\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| \\
& \leq |\gamma_{n+1}(a_I) - \gamma_{n+1}(a_{I'})| + |\gamma_{n+1}(b_I) - \gamma_{n+1}(b_{I'})| \\
& \leq \ell(\gamma_{n+1}|_{I \setminus I'}) = \ell(\gamma_{n+1}|_I) - \ell(\gamma_{n+1}|_{I'}) \\
& \stackrel{(2.6), (3.18)}{\leq} (1 + \varepsilon\beta)\ell(\gamma_n(I)) - (1 - C\beta - 2\varepsilon\beta)\text{diam}Q \\
& \leq (1 + \varepsilon\beta)\text{diam}Q - (1 - C\beta - 2\varepsilon\beta)\text{diam}Q \\
& = (3\varepsilon\beta + C\beta)\text{diam}Q \tag{3.19}
\end{aligned}$$

$$\stackrel{(3.13)}{\leq} \frac{3\varepsilon\beta + C\beta}{4} |\gamma_{n+1}(a_I) - \gamma_{n+1}(b_I)| \tag{3.20}$$

Hence

$$\begin{aligned}
|\gamma_{n+1}(a_I) - \gamma_{n+1}(b_I)| & \leq \frac{|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})|}{1 - \frac{3\varepsilon\beta + C\beta}{4}} \\
& \leq 2|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| \tag{3.21}
\end{aligned}$$

if we pick $\varepsilon < \frac{1}{6}$ and $\beta < \frac{C}{2}$. By Lemma 11

$$\begin{aligned}
& \ell(\gamma_{n+1}|_{I'}) - |\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| \\
& = \ell(\gamma_{n+1}|_I) - |\gamma_{n+1}(a_I) - \gamma_{n+1}(b_I)| \stackrel{(3.14)}{<} \varepsilon\beta |\gamma_{n+1}(a_I) - \gamma_{n+1}(b_I)| \\
& \stackrel{(3.21)}{\leq} 2\varepsilon\beta |\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})|
\end{aligned}$$

which proves (3.16). \square

By the main assumption in Theorem 4, and because we're assuming $n = 0$ so that $M^{-n} = 1 < r_0$,

$$\begin{aligned}
& \beta < \beta'(x_Q, (1 - C\beta)cM^{-N}) \\
& \leq \frac{\ell(\gamma_{n+1}|_{I'}) - |\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| + \sup_{z \in (1-C\beta)B_Q} d(z, \gamma_{n+1}(I'))}{|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})|} \\
& \stackrel{(3.16)}{\leq} \frac{2\varepsilon\beta |\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| + \sup_{z \in (1-C\beta)B_Q} d(z, \gamma_{n+1}(I'))}{|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})|} \\
& = 2\varepsilon\beta + \frac{\sup_{z \in (1-C\beta)B_Q} d(z, \gamma_{n+1}(I'))}{|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})|}
\end{aligned}$$

so there is $z \in \Gamma \cap (1 - C\beta)B_Q$ with

$$\begin{aligned}
d(z, \gamma_{n+1}(I')) &\geq (\beta - 2\varepsilon\beta)|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| \\
&\stackrel{(3.21)}{\geq} \frac{\beta - 2\varepsilon\beta}{2}|\gamma_{n+1}(a_I) - \gamma_{n+1}(b_I)| \\
&\stackrel{(3.13)}{\geq} \frac{\beta - 2\varepsilon\beta}{8}\text{diam}Q \geq \frac{\beta}{16}\text{diam}Q
\end{aligned} \tag{3.22}$$

if $\varepsilon < \frac{1}{4}$ (see Figure 4).

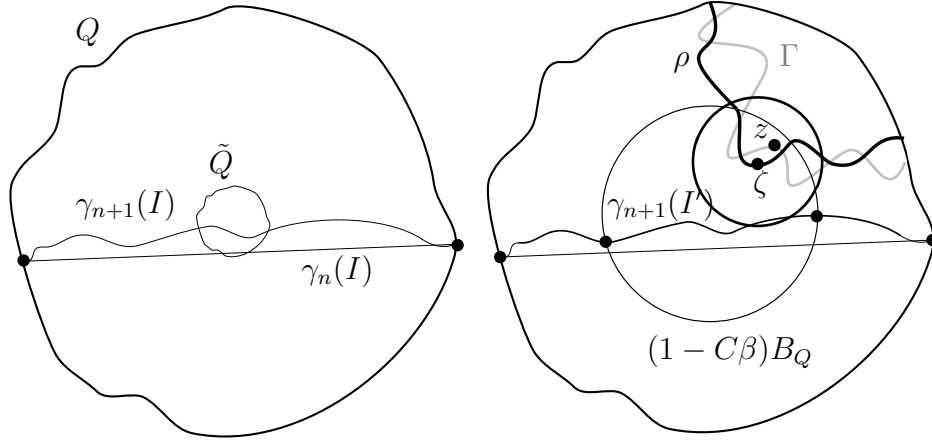


Figure 4

Since $\gamma_{n+1}([0, 1])$ hits every cube in $\mathcal{L}_1(Q)$, which all have diameter at most $2(1 + \varepsilon\beta)cM^{-N-1}$ (recall N was chosen so that $Q \in \Delta_N$),

$$\Gamma_{n_0} \cap Q \subseteq (\gamma_{n+1}(I))_{2(1+\varepsilon\beta)cM^{-N-1}} \subseteq (\gamma_{n+1}(I))_{4cM^{-N-1}}$$

Observe that since $Q \in \tilde{\mathcal{L}}_n$, we have $N < n_0$. Since $\tilde{X}_{n_0} \subseteq \Gamma_{n_0} \cap \Gamma$ and $N < n_0$,

$$\begin{aligned}
\Gamma \cap (1 - C\beta)Q &\subseteq \Gamma \cap Q_0 \subseteq (\Gamma_{n_0} \cap Q)_{2M^{-n_0}} \\
&\subseteq (\gamma_{n+1}(I))_{4cM^{-N-1} + 2M^{-n_0}} \\
&\subseteq (\gamma_{n+1}(I))_{(4cM^{-N-1} + 2M^{-1}M^{-N})} \\
&= (\gamma_{n+1}(I))_{(2 + \frac{1}{c})M^{-1}\text{diam}B_Q} \\
&\subseteq (\gamma_{n+1}(I))_{\frac{2}{c}M^{-1}\text{diam}B_Q}
\end{aligned}$$

since $c < \frac{1}{8}$. Hence there is $t \in [0, 1]$ such that

$$|\gamma_{n+1}(t) - z| < \frac{2}{c} M^{-1} \text{diam} B_Q \leq \frac{\varepsilon \beta}{4c} \text{diam} Q \quad (3.23)$$

and so

$$\begin{aligned} d(\gamma_{n+1}(t), \gamma_{n+1}(I')) &\geq d(z, \gamma_{n+1}(I')) - |\gamma_{n+1}(t) - z| \\ &\stackrel{(3.22), (3.23)}{\geq} \left(\frac{\beta}{16} - \frac{\varepsilon \beta}{4c} \right) \text{diam} Q \geq \frac{\beta}{32} \text{diam} Q \end{aligned} \quad (3.24)$$

for $\varepsilon < \frac{c}{8}$. Also, since $z \in (1 - C\beta)B_Q$, we know that

$$\begin{aligned} B_Q &\supseteq B \left(z, \frac{C\beta}{2} \text{diam} B_Q \right) \stackrel{(2.6)}{\supseteq} B \left(z, \frac{C\beta}{2(1 + \varepsilon\beta)} \text{diam} Q \right) \\ &\supseteq B \left(z, \frac{C\beta}{4} \text{diam} Q \right) \\ &\stackrel{(3.23)}{\supseteq} B \left(\gamma_{n+1}(t), \left(\frac{C\beta}{4} - \frac{\varepsilon\beta}{4c} \right) \text{diam} Q \right) \\ &\supseteq B \left(\gamma_{n+1}(t), \frac{C\beta}{8} \text{diam} Q \right) \end{aligned} \quad (3.25)$$

for $\varepsilon < \frac{Cc}{2}$. In particular, $t \in \gamma^{-1}(B_Q)$. Note

$$\begin{aligned} d(\gamma_{n+1}(t), \gamma_{n+1}(I)) &\geq d(\gamma_{n+1}(t), \gamma_{n+1}(I')) - \max\{\text{diam} \gamma([a_I, a'_I]), \text{diam} \gamma([b'_I, b_I])\} \\ &\geq d(\gamma_{n+1}(t), \gamma_{n+1}(I')) - \ell(\gamma|_{I/I'}) \\ &\stackrel{(3.19)}{\geq} d(\gamma_{n+1}(t), \gamma_{n+1}(I')) - (3\varepsilon\beta + C\beta) \text{diam} Q \\ &\stackrel{(3.24)}{\geq} \frac{\beta}{32} \text{diam} Q - (3\varepsilon\beta + C\beta) \text{diam} Q \geq \frac{\beta}{64} \text{diam} Q \end{aligned} \quad (3.26)$$

for $\varepsilon < \frac{1}{384}$ and $C < \frac{1}{128}$. Thus, since of course $\frac{C}{8} < \frac{1}{128}$, we have

$$B \left(\gamma_{n+1}(t), \frac{C\beta}{8} \text{diam} Q \right) \subseteq Q \setminus (\gamma_{n+1}(I))_{\frac{\beta}{128} \text{diam} Q}$$

Observe that if $\gamma_{n+1}(t) \notin \Gamma_{n_0}$, then $t \in (a, b)$ for some $[a, b] \in \lambda(Q)$, where $\gamma_{n+1}(a)$ and $\gamma_{n+1}(b)$ are both in Γ_{n_0} . Moreover, $\gamma_{n+1}((a, b))$ is contained in a cube $R \in \mathcal{L}_1(Q)$, which has diameter at most $2(1 + \varepsilon\beta)cM^{-N-1}$, hence we know there is

$$\begin{aligned}
\zeta &\in \Gamma_{n_0} \cap B(\gamma_{n+1}(t), 2(1 + \varepsilon\beta)cM^{-N-1}) \\
&\subseteq B(\gamma_{n+1}(t), (1 + \varepsilon\beta)M^{-1}\text{diam}Q) \\
&= B\left(\gamma_{n+1}(t), (1 + \varepsilon\beta)\frac{\varepsilon\beta}{8}\text{diam}Q\right) \\
&\subseteq B\left(\gamma_{n+1}(t), \frac{\varepsilon\beta}{4}\text{diam}Q\right) \\
&\subseteq B\left(\gamma_{n+1}(t), \frac{C\beta}{16}\text{diam}Q\right)
\end{aligned} \tag{3.27}$$

for $\varepsilon < \frac{C}{4}$, and so

$$B\left(\zeta, \frac{C\beta}{16}\text{diam}Q\right) \subseteq B\left(\gamma_{n+1}(t), \frac{C\beta}{8}\text{diam}Q\right) \subseteq Q \setminus (\gamma_{n+1}(I))_{\frac{\beta}{128}\text{diam}Q}. \tag{3.28}$$

Thus, since Γ_{n_0} is connected and $\text{diam}\Gamma_{n_0} > \text{diam}Q_0 > \frac{C\beta}{16}\text{diam}Q$, we know there is a curve $\rho \subseteq \Gamma_{n_0} \cap B(\zeta, \frac{C\beta}{16}\text{diam}Q)$ connecting ζ to $B(\zeta, \frac{C\beta}{16}\text{diam}Q)^c$, and hence has diameter at least $\frac{C\beta}{16}\text{diam}Q$. Hence,

$$\mathcal{H}_\infty^1(\rho) \geq \text{diam}\rho \geq \frac{C\beta}{16}\text{diam}Q.$$

Moreover,

$$\mathcal{H}_\infty^1(\gamma(I)) \geq \text{diam}\gamma(I) \geq |\gamma(a_I) - \gamma(b_I)| \stackrel{(3.15)}{\geq} (1 - 3\varepsilon\beta)\text{diam}Q$$

Thus, since the cubes in $\mathcal{L}_1(Q)$ intersecting ρ are disjoint from those intersecting $\gamma(I)$ (by (3.28) if $\varepsilon < \frac{1}{128}$), we have

$$\mathcal{H}_\infty^1(Q) \geq \frac{C\beta}{16}\text{diam}Q + (1 - 3\varepsilon\beta)\text{diam}Q \geq \left(1 + \frac{C\beta}{32}\right)\text{diam}Q$$

for $\varepsilon < \frac{C}{96}$. Hence, by picking $K = \frac{C}{32}$, we see that $Q \in \Delta_{Bad}$, which finishes the proof of Lemma 14

3.4 Geometric martingales and the proof of Lemma 15

Define $k(Q)$ to be the number of cubes in Δ_{Bad} that properly contain Q , and set

$$\begin{aligned}\Delta_{Bad,j} &= \{Q \in \Delta_{Bad} : k(Q) = j\}. \\ Bad_j(Q) &= \{R \subseteq Q : k(R) = k(Q) + j\}. \\ G(Q) &= (\Gamma_{n_0} \cap Q) \setminus \bigcup_{R \in Bad_1(Q)} R.\end{aligned}$$

We'll define a nonnegative function $w : \Gamma_{n_0} \rightarrow [0, \infty)$ in a martingale fashion by declaring the value of

$$w_Q^j(A) := \int_{\Gamma_{n_0} \cap A} w_Q^j d\mathcal{H}^1$$

on various parts of its support for a sequence of functions w_Q^j and define $w_Q = \lim_j w_Q^j$. First set

$$w_Q^0(Q) = \text{diam}Q, w_Q^0|_{Q^c} \equiv 0 \quad (3.29)$$

and construct w_Q^{j+1} from w_Q^j as follows:

1. If $R \in Bad_j(Q)$ for some j , and $S \in Bad_1(R)$, set w_Q^{j+1} to be constant in S so that

$$w_Q^{j+1}(S) = w_Q^j(R) \frac{\text{diam}S}{\sum_{T \in Bad_1(R)} \text{diam}T + \mathcal{H}^1(G(R))}.$$

2. Set w_Q^{j+1} to be constant in $G(R)$ so that

$$w_Q^{j+1}(G(R)) = w_Q^j(R) - \sum_{S \in Bad_1(R)} w_Q^j(S).$$

3. For points x not contained in any $R \in Bad_j(Q)$, set $w_Q^{j+1}(x) = w_Q^j(x)$.

We will need the following inequality:

$$\sum_{T \in Bad_1(R)} \text{diam}T + \mathcal{H}^1(G(R)) \geq \mathcal{H}_\infty^1(R \cap \Gamma_{n_0}) \geq (1 + K\beta) \text{diam}R. \quad (3.30)$$

The first inequality comes from the fact that if $\delta > 0$ and A_i is a cover of $G(R)$ by sets so that $\sum \text{diam} A_i < \mathcal{H}^1(G(R)) + \delta$, then $\{A_i\} \cup \text{Bad}_1(R)$ is a cover of R (up to a set of \mathcal{H}^1 -measure zero by Lemma 10), and so

$$\sum_{T \in \text{Bad}_1(R)} \text{diam} T + \mathcal{H}^1(G(R)) + \delta > \sum \text{diam} A_i + \sum_{T \in \text{Bad}_1(R)} \text{diam} T \geq \mathcal{H}_\infty^1(R \cap \Gamma_{n_0})$$

which gives the first inequality in (3.30) by taking $\delta \rightarrow 0$. The last inequality in (3.30) is from the definition of Δ_{Bad} .

For $S \in \text{Bad}_1(R)$ and $R \in \text{Bad}_j(Q)$,

$$\begin{aligned} \frac{w_Q^{j+1}(S)}{\text{diam} S} &= \frac{w_Q^j(R)}{\sum_{T \in \text{Bad}_1(R)} \text{diam} T + \mathcal{H}^1(G(R))} \stackrel{(3.30)}{\leq} \frac{w_Q^j(R)}{\text{diam} R} \frac{1}{1 + K\varepsilon\beta} \\ &\leq \frac{w_Q^0(Q)}{\text{diam} Q} \frac{1}{(1 + K\varepsilon\beta)^{j+1}} \stackrel{(3.29)}{=} \frac{1}{(1 + K\varepsilon\beta)^{j+1}} \end{aligned} \quad (3.31)$$

Hence, since w_Q^{j+1} is constant in S , for $x \in S$,

$$\begin{aligned} w_Q^{j+1}(x) &= w_Q^j(R) \frac{\text{diam} S}{\sum_{T \in \text{Bad}_1(R)} \text{diam} T + \mathcal{H}^1(G(R))} \frac{1}{\mathcal{H}^1(S \cap \Gamma_{n_0})} \\ &\stackrel{(3.9)}{\leq} w_Q^j(R) \frac{1}{\sum_{T \in \text{Bad}_1(R)} \text{diam} T + \mathcal{H}^1(G(R))} \frac{1}{1 + K\beta} \\ &\stackrel{(3.30)}{\leq} \frac{w_Q^j(R)}{\text{diam} R} \frac{1}{(1 + K\varepsilon\beta)^2} \\ &\stackrel{(3.31)}{\leq} \frac{w_Q^0(Q)}{\text{diam} Q} \frac{1}{(1 + K\varepsilon\beta)^{j+2}} \\ &= \frac{1}{(1 + K\varepsilon\beta)^{j+2}}. \end{aligned} \quad (3.32)$$

and if $x \in G(R)$, using the same estimates above

$$\begin{aligned}
w_Q^{j+1}(x) &= w_Q^j(R) \frac{\mathcal{H}^1(G(R))}{\sum_{T \in \text{Bad}_1(R)} \text{diam} T + \mathcal{H}^1(G(R))} \frac{1}{\mathcal{H}^1(G(R))} \\
&= w_Q^j(R) \frac{1}{\sum_{T \in \text{Bad}_1(R)} \text{diam} T + \mathcal{H}^1(G(R))} \\
&\leq \frac{w_Q^j(R)}{\text{diam} R} \frac{1}{1 + K\varepsilon\beta} \\
&\leq \frac{1}{(1 + K\varepsilon\beta)^{j+1}}.
\end{aligned} \tag{3.33}$$

Note that since $\Delta_{\text{Bad}} \subseteq \bigcup_{j=0}^{n_0} \Delta_j$, and $\mathcal{H}^1(\bigcup_{Q \in \Delta} \partial Q) = 0$, almost any point $x \in Q_0 \cap \Gamma_{n_0}$ is contained in at most finitely many cubes in Δ_{Bad} , and hence the value of $w_Q^{j+1}(x)$ changes at most finitely many times in j , thus the limit $w_Q = \lim_j w_Q^j$ is well defined. For $x \in Q \cap \Gamma_{n_0}$, set $k(x) = k(R)$ where $R \subseteq Q$ is the smallest cube in Δ_{Bad} containing x . Then (3.32) and (3.33) imply

$$w_Q(x) \leq \frac{1}{(1 + K\varepsilon\beta)^{k(x) - k(Q)}}$$

and so

$$\sum_{x \in Q \in \Delta_{\text{Bad}}} w_Q(x) \leq \sum_{j=0}^{k(x)} \frac{1}{(1 + K\varepsilon\beta)^j} \leq \sum_{j=0}^{\infty} \frac{1}{(K + \varepsilon\beta)^j} = \frac{1 + K\varepsilon\beta}{K\varepsilon\beta} \leq \frac{2}{\varepsilon\beta}$$

for $\varepsilon < \frac{1}{K}$. Hence,

$$\begin{aligned}
\sum_{Q \in \Delta_{\text{Bad}}} \text{diam} Q &= \sum_{Q \in \Delta_{\text{Bad}}} \int_Q w_Q(x) d\mathcal{H}^1(x) \\
&= \int_{\Gamma_{n_0}} \left(\sum_{x \in Q \in \Delta_{\text{Bad}}} w_Q(x) \right) d\mathcal{H}^1(x) \\
&\leq \frac{2}{K\varepsilon\beta} \mathcal{H}^1(\Gamma_{n_0})
\end{aligned}$$

which finishes the proof of Lemma 15.

4 Antenna-like sets

This section is devoted to the proof of Theorem 6.

It is easy to verify using the definitions that being antenna-like is a quasisymmetric invariant quantitatively, so by Theorem 4, it suffices to verify that any c -antenna ball $B(x, r)$ has $\beta'(x, r) > \frac{c}{7}$.

Let X be a compact connected metric space and $B(x, r)$ have a c -antenna for some $x \in X$ and $r > 0$, so there exists a homeomorphism $h : \bigcup_{i=1}^3 [0, e_i] \rightarrow X \cap B(x, r)$ so that

$$d(h(e_i), h([0, e_j] \cup [0, e_k])) \geq cr \quad (4.1)$$

for all permutations (i, j, k) of $(1, 2, 3)$ (see Figure 5).

Let $s : [0, 1] \rightarrow B(x, r)$ satisfy

$$\ell(s|_{[0,1]}) - |s(0) - s(1)| + \sup_{z \in X \cap B(x,r)} d(z, s([0, 1])) < 2\beta'(x, r)|s_0 - s_1| =: \beta.$$

Set $x_i = h(e_i)$ for $i = 1, 2, 3$ and let

$$t_1 = \inf s^{-1} \left(\bigcup_{i=1}^3 B(x_i, \beta) \right).$$

This always exists since $X \cap B(x, r) \subseteq (s([0, 1]))_\beta$. Without loss of generality, assume

$$s(t_1) \in B(x_1, \beta). \quad (4.2)$$

Similarly, let

$$t_2 = \inf s^{-1} \left(\bigcup_{i=2}^3 B(x_i, \beta) \right) \quad (4.3)$$

and again, without loss of generality, assume $s(t_2) \in B(x_2, \beta)$.

Observe that $h([0, e_1] \cup [0, e_3])$ is a path connecting x_1 to x_3 , where the latter point is not contained in $(s([t_1, t_2]))_\beta$ by our choices of t_1 and t_2 , although the latter point is. Pick a point $z \in h([0, e_1] \cup [0, e_3])$ so that $d(z, s([t_1, t_2])) = \beta$. Pick $\zeta_1 \in [t_1, t_2]$ and $\zeta_2 \in (t_2, 1]$ so that

$$|s(\zeta_1) - z| = d(z, s([t_1, t_2])) = \beta \text{ and } |s(\zeta_2) - z| < \beta. \quad (4.4)$$

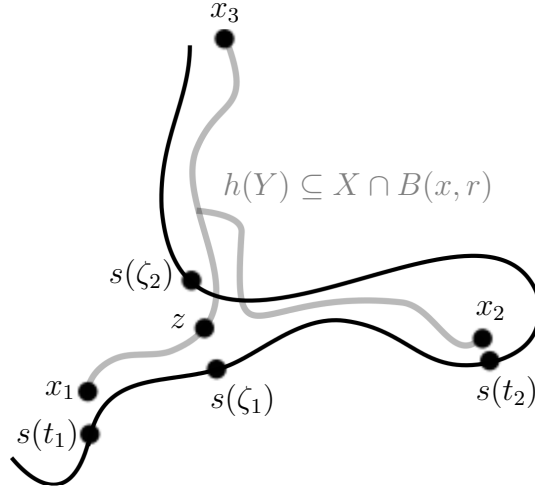


Figure 5

Then by Lemma 11,

$$\begin{aligned}
2\beta'(x, r)|s_0 - s_1| &> \ell(s|_{[0,1]}) - |s(0) - s(1)| \\
&\geq \ell(s|_{[\zeta_1, \zeta_2]}) - |s(\zeta_1) - s(\zeta_2)| \\
&\geq \ell(s|_{[\zeta_1, t_2]}) + \ell(s|_{[t_2, \zeta_2]}) - |s(\zeta_1) - z| - |z - s(\zeta_2)| \\
&\stackrel{(4.4)}{>} |s(\zeta_1) - s(t_2)| + |s(t_2) - s(\zeta_2)| - \beta - \beta \\
&\geq |z - x_2| - |s(\zeta_1) - z| - |x_2 - s(t_2)| \\
&\quad + |x_2 - z| - |s(t_2) - x| - |s(\zeta_2) - z| - 2\beta \\
&\stackrel{(4.1), (4.4)}{\geq} cr - \beta - \beta + cr - \beta - \beta - 2\beta \\
&= 2cr - 6\beta \geq c|s(0) - s(1)| - 12\beta(x, r)|s(0) - s(1)|
\end{aligned}$$

which yields

$$\beta'(x, r) \geq \frac{c}{7}$$

and completes the proof of Theorem 6

5 Comparison of the β -numbers

For quantities A and B , we will write $A \lesssim B$ if there is a universal constant C so that $A \leq CB$, and $A \sim B$ if $A \lesssim B \lesssim A$.

Lemma 18. *Let $X \subseteq \ell^\infty$ be a compact connected set, $x \in X$, and $0 < r < \frac{\text{diam}X}{2}$. Then*

$$\beta'(x, r) \leq \hat{\beta}(x, r) \lesssim \beta'(x, r)^{\frac{1}{2}}. \quad (5.1)$$

Proof. The first inequality follows trivially from the definitions, since each sequence $y_0, \dots, y_n \in X$ induces a finite polygonal Lipschitz path s in ℓ^∞ for which

$$\ell(s) - |s(0) - s(1)| = \sum_{i=0}^{n-1} |y_i - y_{i+1}| - |y_0 - y_n|.$$

For the opposite inequality, let $s : [0, 1] \rightarrow \ell^\infty$ be such that

$$\frac{\ell(s) - |s(0) - s(1)| + \sup_{z \in B(x, r)} d(z, s([0, 1]))}{|s(0) - s(1)|} \leq 2\beta'(x, r) =: \beta. \quad (5.2)$$

Let

$$A = s^{-1}((s([0, 1]))_{2\beta|s(0) - s(1)|})$$

which is a relatively open subset of $[0, 1]$. Let $a = \inf A$ and define $a = t_0 < t_1 < \dots < t_n \leq 1$ inductively by setting

$$t_{i+1} = \inf\{t \in A \cap (t_i, b] : d(s(t), s([t_0, t_i])) > \beta^{\frac{1}{2}}|s(0) - s(1)|\}.$$

We claim that

$$n \sim \beta^{-\frac{1}{2}}|s(0) - s(1)|. \quad (5.3)$$

To see this, observe that since $|s(t_i) - s(t_{i+1})| \geq \beta^{\frac{1}{2}}|s(0) - s(1)|$, the sets $B(s(t_i), \frac{\beta^{\frac{1}{2}}}{2}|s(0) - s(1)|)$ are disjoint, so that

$$n \frac{\beta^{\frac{1}{2}}}{2}|s(0) - s(1)| \leq \ell(s) \leq (1 + \beta)|s(0) - s(1)| \leq 2|s(0) - s(1)|$$

which gives $n \leq 4\beta^{-\frac{1}{2}}$. On the other hand, the balls $B(s(t_i), 2\beta^{\frac{1}{2}}|s(0) - s(1)|)$ cover $s([0, 1])$, and so

$$\begin{aligned} |s(0) - s(1)| &\leq \ell(s) \leq \sum_{i=0}^n \text{diam} B(s(t_i), 2\beta^{\frac{1}{2}}|s(0) - s(1)|) \\ &\leq (n+1)4\beta^{\frac{1}{2}}|s(0) - s(1)| \leq 8n\beta^{\frac{1}{2}}|s(0) - s(1)| \end{aligned}$$

which gives $n \geq (8\beta)^{-1}$, and this proves (5.3).

By the definition of A , there are

$$y_i \in \overline{B(s(t_i), 2\beta|s(0) - s(1)|)}.$$

Then

$$\begin{aligned} \sum_{i=0}^{n-1} |y_i - y_{i+1}| - |y_0 - y_1| &\leq \sum_{i=0}^{n-1} |s(t_i) - s(t_{i+1})| + 4n\beta|s(0) - s(1)| - |s(t_0) - s(t_n)| \\ &\stackrel{(5.3)}{\leq} \ell(s|_{[t_0, t_n]}) - |s(t_0) - s(t_n)| + C\beta^{\frac{1}{2}}|s(0) - s(1)| \\ &\stackrel{(5.2)}{\leq} \beta|s_0 - s_1| + C\beta^{\frac{1}{2}}|s(0) - s(1)| \lesssim \beta^{\frac{1}{2}}|s(0) - s(1)|. \end{aligned}$$

Claim: $|s(0) - s(1)| \lesssim |s(t_0) - s(t_n)|$.

Since X is connected and $r < \frac{\text{diam} X}{2}$, there is a path connecting x to $B(x, r)^c$, which naturally must be of diameter at least r , hence

$$|s(0) - s(1)| \leq 2r \leq 2(\ell(s|_{[t_0, t_n]}) - 4\beta|s_0 - s_1|) \leq 2|s(t_0) - s(t_n)| + C\beta^{\frac{1}{2}}|s(0) - s(1)|,$$

which, if $\beta^{\frac{1}{2}}$ is small enough, this implies

$$|s(0) - s(1)| \leq 4|s(t_0) - s(t_n)| = 4|y_0 - y_n|$$

so that the above estimates imply

$$\sum_i |y_i - y_{i+1}| - |y_0 - y_n| \lesssim \beta^{\frac{1}{2}}|s(0) - s(1)| \leq 4\beta^{\frac{1}{2}}|y_0 - y_n| \quad (5.4)$$

Moreover,

$$\begin{aligned} X \cap B(x, r) &\subseteq (s([0, 1]))_{\beta|s(0) - s(1)|} \subseteq \bigcup_i B(s(t_i), (2\beta^{\frac{1}{2}} + \beta)|s(0) - s(1)|) \\ &\subseteq \bigcup_i B(y_i, (2\beta^{\frac{1}{2}} + \beta + 2\beta)|s(0) - s(1)|) \\ &\subseteq \bigcup_i B(y_i, 5\beta^{\frac{1}{2}}|s(0) - s(1)|) \\ &\subseteq \bigcup_i B(y_i, 20\beta^{\frac{1}{2}}|y_0 - y_n|) \end{aligned} \quad (5.5)$$

Thus (5.4) and (5.5) imply

$$\hat{\beta}(x, r) \leq 20\beta^{\frac{1}{2}} = 20\sqrt{2}\beta'(x, r)^{\frac{1}{2}}.$$

□

Proposition 19. *If X is a compact connected subset of some Hilbert space, then for $x \in X$ and $r < \frac{\text{diam}X}{2}$,*

$$\beta''(x, r) \leq \beta(x, r) \lesssim \beta''(x, r)$$

where

$$\beta''(x, r) = \inf_s \left(\left(\frac{\ell(s) - |s(0) - s(1)|}{|s(0) - s(1)|} \right)^{\frac{1}{2}} + \frac{\sup_{z \in B(x, r)} d(z, s([0, 1]))}{|s(0) - s(1)|} \right).$$

In particular,

$$\beta'(x, r) \leq \beta(x, r) \lesssim \beta'(x, r)^{\frac{1}{2}}. \quad (5.6)$$

We quickly note that (5.6) is tight in the sense that if $X \subseteq \mathbb{C}$, $0 \in X$, and $B(0, 1) \cap X = [-1, 1] \cup [0, i\varepsilon]$, then by Theorem 6 and (5.6), for all $\varepsilon > 0$,

$$\beta(0, 1) \leq \varepsilon \leq 7\beta'(0, 1) \leq 7\beta(0, 1) \leq 7\varepsilon.$$

However, if $X \cap B(x, r) = [-1, 0] \cup [0, e^{i\varepsilon}]$, then for all $\varepsilon > 0$, again by (5.6) (and estimating $\beta''(0, 1)$ by letting s be the path traversing the segments $[-1, 0] \cup [0, e^{i\varepsilon}]$),

$$\beta(0, 1)^2 \sim \varepsilon^2 \gtrsim \beta'(0, 1) \gtrsim \beta(0, 1)^2.$$

Proof. For the first inequality, simply let $s : [0, 1] \rightarrow \mathcal{H}$ be the line segment spanning $L \cap B(x, r)$ where L is some line passing through $B(x, \frac{r}{2})$. Then $\ell(s) = \mathcal{H}^1(L \cap B(x, r)) \geq r$ and hence

$$\beta''(x, r) \leq \frac{\sup_{z \in B(x, r)} d(z, s([0, 1]))}{|s(0) - s(1)|} \leq \frac{\sup_{z \in B(x, r)} d(z, L)}{r}.$$

Observe that since $x \in X$, the range of admissible lines in the infimum in (1.1) can be taken to be lines intersecting $B(x, \frac{r}{2})$. Using this fact and infimizing the above inequality over all such lines proves the first inequality in (5.6).

For the opposite inequality, let s satisfy

$$\left(\frac{\ell(s) - |s(0) - s(1)|}{|s(0) - s(1)|} \right)^{\frac{1}{2}} + \frac{\sup_{z \in B(x, r)} d(z, s([0, 1]))}{|s(0) - s(1)|} \leq 2\beta''(B(x, r)) =: \beta.$$

Let

$$\beta(s) := \sup_{t \in [0, 1]} d(s(t), [s(0), s(1)]).$$

Then by the Pythagorean theorem, there is $c > 0$ so that

$$(1 + c\beta(s)^2)|s(0) - s(1)| \leq \ell(s) \leq (1 + \beta^2)|s(0) - s(1)|$$

so that

$$\beta(s) \leq c^{-1}\beta,$$

and hence, if L is the line passing through $s(0)$ and $s(1)$,

$$\begin{aligned} \beta(x, r) &\leq \sup_{z \in B(x, r) \cap X} d(z, L) \leq \sup_{z \in B(x, r) \cap X} d(z, [s(0), s(1)]) \\ &\leq \beta(s) + \sup_{z \in B(x, r) \cap X} d(z, s([0, 1])) \leq c^{-1}\beta + \beta \lesssim \beta \end{aligned}$$

□

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